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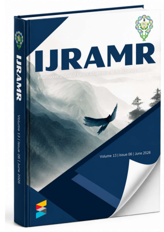

REVIEW ARTICLE

On homogeneous cubic equation with four unknowns $x^3 + y^3 = 4zw^2$

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ABSTRACT

This paper concerns with the problem of obtaining non-zero distinct integer solutions to homogeneous cubic equation with four unknowns given by $x^3 + y^3 = 4zw^2$.A few interesting properties among the solutions are presented. 2020 Mathematical science classification: 11D25

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INTRODUCTION

The cubic diophantine equations are rich in variety and offer an unlimited field for research. In particular, refer [1-27] for a few problems on cubic equation with many unknowns. This paper concerns with yet another interesting homogeneous cubic diophantine equation with four unknowns given by $x^3 + y^3 = 4zw^2$ for determining its infinitely many non-zero distinct integral solutions through employing linear transformations and factorization method. A few interesting relations among the solutions are presented.

Method of analysis

Thehomogeneous cubic equation with four unknowns to be solved is represented by

$$x^3 + y^3 = 4zw^2. \tag{1.1}$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 2u \tag{1.2}$$

in (1.1) leads to

$$u^2 + 3v^2 = 4w^2. \tag{1.3}$$

Different methods of obtaining the patterns of integer solutions to (1.1) are illustrated below:

Patterns

Pattern 2.1

Let

$$w=a^2+3b^2, \tag{2.1}$$

where a and b are non-zero integers.

Write 4 as

$$4=(1+i\sqrt{3})(1-i\sqrt{3}). \tag{2.2}$$

Using (2.1), (2.2) in (1.3) and applying the method of factorization, define

$$(u+i\sqrt{3}v)=(1+i\sqrt{3})(a+i\sqrt{3}b)^2, \tag{2.3}$$

from which , we have

$$\left.\begin{array}{l} u=a^2-6ab-3b^2 \\ v=a^2+2ab-3b^2 \end{array}\right\}. \tag{2.4}$$

Using (2.4) and (1.2), the values of x, y and z are given by

$$\left.\begin{array}{l} x=x(a,b)=2a^2-4ab-6b^2 \\ y=y(a,b)=-8ab \\ z=z(a,b)=2a^2-12ab-6b^2 \end{array}\right\}. \tag{2.5}$$

Thus (2.1) and (2.5) represent the non-zero integer solutions to (1.1).

Observations 2.1

1. $32w^2-2(2x-y)^2$ is a square multiple of six
2. Each of the following two expressions is a perfect6 square

   $48w^2-3(2x-y)^2$, $(z-x)y$
3. $x^3+y^3-z^3=-3xyz$
4. The triple $(x, z, 2(x+z))$ is such that the product of any two members of the triple added with $(4ab)^2$ is a perfect square
5. $x-z$ is a perfect square when $a=2b$
6. $x-z=16t_{3,b}$ when $a=b+1$
7. $x-z=48P_b^3$ when $a=(b+1)(b+2)$
8. $x-z=16P_b^5$ when $a=b(b+1)$

Pattern 2.2

Write (1.3) as

$$u^2 + 3v^2 = 4w^2 * 1. \tag{2.6}$$

Write 1 as

$$1 = \left(\frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \right). \tag{2.7}$$

Using (2.1), (2.2),(2.7) in (2.6) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (1 + i\sqrt{3})(a + i\sqrt{3}b)^2 \left(\frac{1 + i\sqrt{3}}{2} \right), \tag{2.8}$$

from which , we have

$$\left. \begin{aligned} u &= (-a^2 - 6ab + 3b^2) \\ v &= (a^2 - 2ab - 3b^2) \end{aligned} \right\}. \tag{2.9}$$

Using (2.9) and (1.2), the values of x, y and z are given by

$$\left. \begin{aligned} x &= x(a,b) = -8ab \\ y &= y(a,b) = -2a^2 - 4ab + 6b^2 \\ z &= z(a,b) = -2a^2 - 12ab + 6b^2 \end{aligned} \right\}. \tag{2.10}$$

Thus (2.1) and (2.10) represent the non-zero integer solutions to (1.1).

Note 2.2

In addition to (2.7), the integer 1 has the following representations:

$$1 = \frac{(r^2 - 3s^2 + i\sqrt{3}(2rs))(r^2 - 3s^2 - i\sqrt{3}(2rs))}{(r^2 + 3s^2)^2}$$

$$1 = \frac{(3r^2 - s^2 + i\sqrt{3}(2rs))(3r^2 - s^2 - i\sqrt{3}(2rs))}{(3r^2 + s^2)^2}$$

$$1 = \frac{(6s^2 - 6s + 1 + i\sqrt{3}(2s - 1))(6s^2 - 6s + 1 - i\sqrt{3}(2s - 1))}{(6s^2 - 6s + 2)^2}$$

$$1 = \frac{(2s^2 - 2s - 1 + i\sqrt{3}(2s - 1))(2s^2 - 2s - 1 - i\sqrt{3}(2s - 1))}{(2s^2 - 2s + 2)^2}$$

$$1 = \frac{(3s^2 - 1 + i\sqrt{3}(2s))(3s^2 - 1 - i\sqrt{3}(2s))}{(3s^2 + 1)^2}$$

$$1 = \frac{(s^2 - 3 + i\sqrt{3}(2s))(s^2 - 3 - i\sqrt{3}(2s))}{(s^2 + 3)^2}$$

Following the above procedure, one obtains more patterns of integer solutions to (1.1).

Pattern 2.3

By inspection , observe that (1.3) is satisfied by

$$v = 8rs, u = 12r^2 - 4s^2, w = 6r^2 + 2s^2 \tag{2.11}$$

In view of (1.2) , we have

$$x=12r^2-4s^2+8rs$$
$$y=12r^2-4s^2-8rs \qquad (2.12)$$
$$z=24r^2-8s^2$$

Thus, the values of x,y,z,w given by (2.12) & (2.11) satisfy (1.1).

Pattern 2.4

(1.3) is rewritten as

$$u^2=4w^2-3v^2. \qquad (2.13)$$

In (2.13), taking

$$w=X+3T,\ v=X+4T \qquad (2.14)$$

it leads to

$$X^2=u^2+12T^2 \qquad (2.15)$$

which is satisfied by

$$T=2rs,\ u=12r^2-s^2,\ X=12r^2+s^2$$

From (2.14) , one has

$$w=12r^2+s^2+6rs,\ v=12r^2+s^2+8rs \qquad (2.16)$$

In view of (1.2) , we have

$$x=24r^2+8rs$$
$$y=-2s^2-8rs \qquad (2.17)$$
$$z=24r^2-2s^2$$

Thus, the values of x,y,z,w given by (2.17) & (2.16) satisfy (1.1).

Also , (2.15) is written as the system of double equations as shown in Table 1:

Table 1.Systemof DoubleEquations

System	I	II	III	IV	V	VI	VII
$X+u$	T^2	$2T^2$	$3T^2$	$6T^2$	$12T$	$6T$	$4T$
$X-u$	12	6	4	2	T	$2T$	$3T$

Consider system I in Table 1 :Solving the pair of equations, note that

$$X=\frac{T^2+12}{2},u=\frac{T^2-12}{2}.$$

The choice

$$T=2k \qquad (2.18)$$

gives

$$\left.\begin{aligned} X &= 2k^2 + 6 \\ u &= 2k^2 - 6 \end{aligned}\right\}. \quad (2.19)$$

The substitution of (2.18) and (2.19) in (2.14) gives

$$w = \left(2k^2 + 6k + 6\right), v = \left(2k^2 + 8k + 6\right). \quad (2.20)$$

In view of (1.2) , one obtains

$$\left.\begin{aligned} x &= \left(4k^2 + 8k - 2\right) \\ y &= \left(-12 - 8k\right) \\ z &= \left(4k^2 - 12\right) \end{aligned}\right\}. \quad (2.21)$$

Thus (2.21) and (2.20) represent the non-zero integer solutions to (1.1). Consider system II in Table 1: Solving the pair of equations, note that

$$X = T^2 + 3, u = T^2 - 3.$$

Using (2.14), one has

$$\left.\begin{aligned} w &= T^2 + 3T + 3 \\ v &= T^2 + 4T + 3 \end{aligned}\right\}. \quad (2.22)$$

In view of (1.2) one obtains

$$\left.\begin{aligned} x &= \left(2T^2 + 4T + 8\right) \\ y &= \left(-6 - 4T\right) \\ z &= \left(2T^2 - 6\right) \end{aligned}\right\}. \quad (2.23)$$

Thus (2.23) and (2.22) represent the non-zero integer solutions to (1.1). Consider system III in Table 1:Solving the pair of equations, note that

$$X = \frac{3T^2 + 4}{2}, u = \frac{3T^2 - 4}{2}.$$

Using (2.18) , the above equations become

$$\left.\begin{aligned} X &= 6k^2 + 2 \\ u &= 6k^2 - 2 \end{aligned}\right\}. \quad (2.24)$$

The substitution of (2.24) and (2.18) in (2.14) gives

$$\begin{aligned} w &= (6k^2 + 6k + 2), \\ v &= (6k^2 + 8k + 2) \end{aligned} \quad (2.25)$$

In view of (1.2), one obtains

$$\left.\begin{aligned} x &= \left(12k^2 + 8k + 16\right) \\ y &= \left(-4 - 8k\right) \\ z &= \left(12k^2 - 4\right) \end{aligned}\right\}. \quad (2.26)$$

Thus (2.26) and (2.25) represent the non-zero integer solutions to (1.1).

Consider system IV in Table 1:On solving, it is seen that $X = 3T^2 + 1, u = 3T^2 - 1$.

In view of (2.14), we have

$$w = 3T^2 + 3T + 1, v = 3T^2 + 4T + 1. \tag{2.27}$$

Substituting the above values of u and v in (1.2), we get

$$x = 6T^2 + 4T, y = -2 - 4T, z = 6T^2 - 2. \tag{2.28}$$

Thus (2.27) and (2.28) represent the non-zero integer solutions to (1.1). Consider system V in Table 1.On solving,it is seen that

$$T = 2s, X = 13s, u = 11s$$

In view of (2.14),we have

$$w = 19s, v = 21s \tag{2.29}$$

From (1.2), one has

$$x = 32s, y = -10s, z = 22s \tag{2.30}$$

Thus, (2.29) and (2.30) satisfy (1.1). Consider system VI in Table 1.On solving,it is seen that

$$X = 4T, u = 2T$$

In view of (2.14),we have

$$w = 7T, v = 8T \tag{2.31}$$

From (1.2), one has

$$x = 10T, y = -6T, z = 4T \tag{2.32}$$

Thus, (2.31) and (2.32) satisfy (1.1).

Consider system VII in Table 1.On solving,it is seen that

$$T = 2s, X = 7s, u = s$$

In view of (2.14),we have

$$w = 13s, v = 15s \tag{2.33}$$

From (1.2), one has

$$x = 16s, y = -14s, z = 2s \tag{2.34}$$

Thus, (2.33) and (2.34) satisfy (1.1).

Pattern 2.5

Rewrite (1.3) in the ratio form as

$$\frac{u+w}{w+v} = \frac{3(w-v)}{u-w} = \frac{P}{Q}, Q \neq 0 \tag{2.35}$$

Solving (2.35) by the method of cross-multiplication, we obtain

$$u = P^2 + 6PQ - 3Q^2$$
$$v = -P^2 + 2PQ + 3Q^2 \quad (2.36)$$
$$w = P^2 + 3Q^2$$

In view of (1.2), we have

$$x = 8PQ$$
$$y = 2P^2 + 4PQ - 6Q^2 \quad (2.37)$$
$$z = 2P^2 + 12PQ - 6Q^2$$

Thus, the values of x,y,z,w given by (2.37) and (2.36) satisfy (1.1).

CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns. As the cubic equations are rich in variety, one may search for other forms of cubic equations with multivariables to obtain their corresponding solutions .

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